1. Calculus of variation

Definition 1. We define a symmetric positive mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(\vec{x}) = \begin{cases} c_n \exp\left(-\frac{1}{1-\|\vec{x}\|^2}\right), & \text{if } \|\vec{x}\| < 1 \\ 0, & \text{if } \|\vec{x}\| \geqslant 1, \end{cases} \tag{1}$$

where c_n is the constant satisfying $\int_{\mathbb{R}^n} \eta(\vec{x}) d\vec{x} = 1$.

In addition, given $\epsilon > 0$ we define

$$\eta_\epsilon(\vec{x}) = \epsilon^{-n}\eta(\epsilon^{-1}\vec{x}). \tag{2}$$

Theorem 2. *Let $u : \overline{\Omega} \to \mathbb{R}$ be a smooth function. Suppose that*

$$\int_\Omega \|\nabla u\|^2 dx \leqslant \int_\Omega \|\nabla v\|^2 dx, \tag{3}$$

holds for all smooth functions $v : \overline{\Omega} \to \mathbb{R}$ satisfying $u = v$ on $\partial\Omega$. Then, $\Delta u = 0$ holds in $\overline{\Omega}$.

Proof. Since u is smooth, Δu is continuous. Hence, it is enough to show $\Delta u(\vec{y}) = 0$ at each interior point $y \in \Omega$.

Towards a contradiction, we assume that $\Delta u(\vec{y}) > 0$. Then, there exists some small $\epsilon > 0$ such that $\Delta u(\vec{x}) > 0$ holds for $\vec{x} \in B_\epsilon(\vec{y}) \subset \Omega$. We define $\varphi(\vec{x}) = \eta_\epsilon(\vec{x} - \vec{y})$ and $u_s = u + s\varphi$ for each $s \in \mathbb{R}$. Then, we can define a smooth function $I : \mathbb{R} \to \mathbb{R}$ by

$$I(s) = \int_\Omega \|\nabla u_s\|^2 d\vec{x}. \tag{4}$$

Since u_s is smooth and satisfies $u_s = u$ on $\partial\Omega$, we have $I(0) \leqslant I(s)$ for all $s \in \mathbb{R}$. Thus, $I'(0) = 0$. On the other hand, we can directly calculate

$$I'(s) = \frac{d}{ds}\int_\Omega \|\nabla u\|^2 + 2s\nabla u \cdot \nabla\varphi + s^2\|\nabla\varphi\|^2 d\vec{x} = \int_\Omega 2\nabla u \cdot \nabla\varphi + 2s\|\nabla\varphi\|^2 d\vec{x}. \tag{5}$$

Thus,

$$0 = I'(0) = 2\int_\Omega \nabla u \cdot \nabla\varphi d\vec{x} = 2\int_\Omega \varphi\Delta u d\vec{x}. \tag{6}$$

Hence, by definition of the mollifier η, we have

$$0 = 2\int_{B_\epsilon(\vec{y})} \varphi(\vec{x})\Delta u(\vec{x}) d\vec{x}. \tag{7}$$

However, in the ball $B_\epsilon(\vec{y})$, we know $\varphi > 0$ and $\Delta u > 0$, which contradicts to the equation above. Namely, Δu can not be positive everywhere. In the same manner, Δu can not be negative everywhere, and thus $\Delta u = 0$. $\square$

2. Elliptic equation

Given functions $a_{ij}(\vec{x}), b_i(\vec{x}), c(\vec{x})$ defined over $\overline{\Omega}$, we define a linear differential operator $\mathcal{L}$ by

$$\mathcal{L}u = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\vec{x})u_{ij}(\vec{x}) + \sum_{i=1}^{n} b_i(\vec{x})u_i(\vec{x}) + c(\vec{x})u(\vec{x}). \tag{8}$$

Suppose that there exists two positive constant $0 < \lambda \leqslant \Lambda$ such that

$$\lambda\|\xi\|^2 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\vec{x})\xi_i\xi_j \leqslant \Lambda\|\xi\|^2, \tag{9}$$

holds for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$. Then, we call $\mathcal{L}$ is an uniformly elliptic operator. In addition, given a function $f : \overline{\Omega} \to \mathbb{R}$

$$\mathcal{L}u = f, \tag{10}$$

is called a second order linear (uniformly) elliptic partial differential equation.

Definition 3. Given a vector $v \in \mathbb{R}^n$, we define directional derivatives u_v and u_{vv} by

$$u_v = v \cdot \nabla u = v^T \nabla u, \qquad u_{vv} = v^T(\nabla^2 u)v, \tag{11}$$

where $\nabla^2 u$ is the Hessian matrix.

We recall some fact from linear algebra.

Proposition 4. *Let $A = (a_{ij})$ be a symmetric square matrix. Then, A is diagonalizable and has orthonormal eigenvectors $v_1, \cdots, v_n$ and corresponding real eigenvalues $\lambda_1, \cdots, \lambda_n$. In particular,*

$$A = \sum_i \lambda_k v_k v_k^T. \tag{12}$$

Moreover, we can check

Proposition 5.

$$\sum_{i,j} a_{ij}u_{ij} = \sum_i \lambda_i u_{v_i v_i}. \tag{13}$$

Proof. (12) says $a_{ij} = \sum_k \lambda v_k^i v_k^j$ where $v_k = (v_k^1, \cdots, v_k^n) \in \mathbb{R} \times \cdots \times \mathbb{R}$. Hence,

$$\sum_{i,j} a_{ij}u_{ij} = \sum_{i,j,k} \lambda v_k^i v_k^j u_{ij} = \sum_k \lambda_k u_{v_k v_k}. \tag{14}$$

□

Theorem 6 (Maximum principle)**.** *Let $a_{ij}(\vec{x}), b_i(\vec{x}), c(\vec{x})$ be smooth in $\overline{\Omega}$. Suppose $a_{ij}(\vec{x}) = a_{ji}(\vec{x})$ and $c(\vec{x}) \leqslant 0$ holds for all $\vec{x} \in \overline{\Omega}$. In addition, there exists some positive number $\lambda > 0$ such that $\sum_{i,j} a_{ij}(\vec{x})\xi_i\xi_j \geqslant \lambda|\xi|^2$ holds for all $\vec{x} \in \Omega$ and $\xi \in \mathbb{R}^n$.*

Suppose that a smooth function $u : \overline{\Omega} \to \mathbb{R}$ satisfies $\mathcal{L}u \geqslant 0$ in $\overline{\Omega}$ and $u \leqslant 0$ holds on $\partial\Omega$. Then, $u \leqslant 0$ holds in $\overline{\Omega}$.

Proof. We consider a smooth function $\varphi(\vec{x}) = \exp(\alpha x_1)$ for some large enough α to be determined. Since $a_{11}(\vec{x}) \geqslant \lambda > 0$ for all $\vec{x} \in \overline{\Omega}$, we have

$$\mathcal{L}\varphi = a_{11}\alpha^2 + b_1\alpha + c \geqslant \lambda\alpha^2 + b_1\alpha + c. \tag{15}$$

b_1 and c are continuous and thus bounded. Therefore, there exists some large enough α depending on $\lambda, \max|b_1|, \max|c|$ such that $\mathcal{L}\varphi > 0$.

Now, we fix α and define $K = 1 + \max_{\overline{\Omega}} \varphi$. For each $\epsilon > 0$, we define $w^\epsilon = u + \epsilon(\varphi - K)$, and observe that

$$\mathcal{L}w^\epsilon = \mathcal{L}u + \epsilon\mathcal{L}\varphi > 0, \tag{16}$$

holds in $\overline{\Omega}$ and $w^\epsilon < u \leqslant 0$ holds on $\partial\Omega$.

Towards a contradiction, we assume $w^\epsilon(\vec{x}_0) = \max_{\overline{\Omega}} w^\epsilon > 0$. Then, $\vec{x}_0$ must be an interior point of Ω, and thus we have

$$\mathcal{L}w^\epsilon = \sum_{i=1}^n \sum_{j=1}^n a_{ij}w^\epsilon_{ij} + \sum_{i=1}^n b_i w^\epsilon_i + cw^\epsilon \leqslant cw^\epsilon, \tag{17}$$

at $\vec{x}_0$. Thus, $c \leqslant 0$ and $w^\epsilon(\vec{x}_0) > 0$ imply $\mathcal{L}w^\epsilon \leqslant 0$, which contradicts to $\mathcal{L}w^\epsilon > 0$. Namely, $w^\epsilon \leqslant 0$ holds in $\overline{\Omega}$. Hence, passing $\epsilon \to 0$ yields the desired result. □